Trapezoid type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation

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Trapezoid type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation

Silvestru Sever Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
email: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

DST-NRF Centre of Excellence
in the Mathematical and Statistical Sciences,
School of Computer Science & Applied Mathematics,
University of the Witwatersrand, Private Bag 3,
Johannesburg 2050, South Africa

Abstract. In this paper we establish some trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the g-*mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

1 Introduction

Let (a, b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). Following [18,

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p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ by

$$I_{a+,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) \, dt}{[g(x) - g(t)]^{1-\alpha}}, \ a < x \le b$$

and

$$I_{b-,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) \, dt}{[g(t) - g(x)]^{1-\alpha}}, \ a \le x < b.$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t) \, dt}{(x-t)^{1-\alpha}}, \ a < x \le b$$

and

$$J_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t) \, dt}{(t-x)^{1-\alpha}}, \ a \le x < b,$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [18, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t) \, dt}{t}, \ 0 \le a < x \le b$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t) \, dt}{t}, \ 0 \le a < x < b.$$

One can consider the function $g(t) = -t^{-1}$ and define the "*Harmonic fractional integrals*" by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t) \, dt}{(x-t)^{1-\alpha} t^{\alpha+1}}, \ 0 \le a < x \le b$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t) \, dt}{(t-x)^{1-\alpha} t^{\alpha+1}}, \ 0 \le a < x < b.$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the "β-*Exponential fractional integrals*"

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \frac{\exp(\beta t) f(t) \, dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}}, \ a < x \le b$$

and

$$E^{\alpha}_{b-,\beta}f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b \frac{\exp(\beta t) f(t)\, dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}}, \ a \le x < b.$$

In the recent paper [14] we obtained the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1 *Let* $f : [a, b] \to \mathbb{C}$ *be a function of bounded variation on* $[a, b]$ *and* g *be a strictly increasing function on* (a, b), *having a continuous derivative* g' *on* (a, b). *For any* $x \in (a, b)$ *we have the inequalities*

$$\begin{aligned}
&\left| I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x) - \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \right| \\
&\qquad \le \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_t^x (f)\, dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_x^t (f)\, dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \left[[g(x) - g(a)]^{\alpha} \bigvee_a^x (f) + [g(b) - g(x)]^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left((g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right], \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\left| I^{\alpha}_{x-,g}f(a) + I^{\alpha}_{x+,g}f(b) - \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \right| \\
&\qquad \le \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_t^x (f)\, dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_x^t (f)\, dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\
&\qquad \le \frac{1}{\Gamma(\alpha+1)} \left[[g(x) - g(a)]^{\alpha} \bigvee_a^x (f) + [g(b) - g(x)]^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left((g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]. \end{cases}
\end{aligned}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the g-*mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the *LogMeanExp function*.

The following particular case for g-*mean* is of interest [14].

Corollary 1 *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
&\left|I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b))\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)\bigvee_{M_g(a,b)}^t(f)\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right] \\
&\qquad\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f);
\end{aligned}$$

and

$$\begin{aligned}
&\left|I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b))\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right] \\
&\qquad\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f).
\end{aligned}$$

Remark 1 *If we take in Theorem 1* $x = \frac{a+b}{2}$*, then we obtain similar mid-point inequalities, however the details are not presented here. Some applications for the Hadamard fractional integrals are also provided in* [14].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [16]-[27] and the references therein.

Motivated by the above results, in this paper we establish some trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the g-*mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

2 Some identities

We have:

Lemma 1 *Let* $f : [a, b] \to \mathbb{C}$ *be Lebesgue integrable on* $[a, b]$*,* g *be a strictly increasing function on* (a, b)*, having a continuous derivative* g' *on* (a, b) *and* λ*,* μ *some complex parameters:*

(i) *For any* $x \in (a, b)$ *we have the representation*

$$I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x) = \frac{1}{\Gamma(\alpha+1)}\left(\lambda\left[g(x)-g(a)\right]^{\alpha} + \mu\left[g(b)-g(x)\right]^{\alpha}\right) + \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}} + \int_x^b \frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \tag{1}$$

and

$$I^{\alpha}_{x-,g}f(a) + I^{\alpha}_{x+,g}f(b) = \frac{1}{\Gamma(\alpha+1)}\left(\lambda\left[g(x)-g(a)\right]^{\alpha} + \mu\left[g(b)-g(x)\right]^{\alpha}\right) + \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}} + \int_x^b \frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right]. \tag{2}$$

(ii) *We have*

$$\frac{I^{\alpha}_{b-,g}f(a) + I^{\alpha}_{a+,g}f(b)}{2} = \frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{\lambda+\mu}{2} + \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}} + \int_a^b \frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right]. \tag{3}$$

Proof. (i) We observe that

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)[f(t)-\lambda]\,dt}{[g(x)-g(t)]^{1-\alpha}} \\ & = I_{a+,g}^{\alpha} f(x) - \lambda \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}} \\ & = I_{a+,g}^{\alpha} f(x) - \frac{[g(x)-g(a)]^{\alpha}}{\alpha\Gamma(\alpha)}\lambda = I_{a+,g}^{\alpha} f(x) - \frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \end{aligned} \tag{4}$$

for $a < x \le b$ and, similarly,

$$\frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(x)]^{1-\alpha}} = I_{b-,g}^{\alpha} f(x) - \frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{5}$$

for $a \le x < b$.

If $x \in (a, b)$, then by adding the equalities (4) and (5) we get the representation (1).

By the definition of fractional integrals we have

$$I_{x+,g}^{\alpha} f(b) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)\,dt}{[g(b)-g(t)]^{1-\alpha}}, \quad a \le x < b$$

and

$$I_{x-,g}^{\alpha} f(a) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)\,dt}{[g(t)-g(a)]^{1-\alpha}}, \quad a < x \le b.$$

Then

$$\frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I_{x+,g}^{\alpha} f(b) - \frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{6}$$

for $a \le x < b$ and

$$\frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I_{x-,g}^{\alpha} f(a) - \frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{7}$$

for $a < x \le b$.

If $x \in (a, b)$, then by adding the equalities (6) and (7) we get the representation (1).

If we take $x = b$ in (4) we get

$$\frac{1}{\Gamma(\alpha)} \int_a^b \frac{g'(t)[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I_{a+,g}^{\alpha} f(b) - \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{8}$$

while from $x = a$ in (5) we get

$$\frac{1}{\Gamma(\alpha)} \int_a^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I_{b-,g}^{\alpha} f(a) - \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu. \tag{9}$$

If we add (8) with (9) and divide by 2 we get (3). □

Remark 2 *If we take in* (1) *and* (2) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, *then we get*

$$\begin{aligned} & I_{a+,g}^{\alpha} f(M_g(a,b)) + I_{b-,g}^{\alpha} f(M_g(a,b)) \\ &\qquad = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \left(\frac{\lambda+\mu}{2}\right) \\ & + \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-\lambda]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}} \right] \end{aligned}$$

and

$$\begin{aligned} I_{M_g(a,b)-,g}^{\alpha} f(a) + I_{M_g(a,b)+,g}^{\alpha} f(b) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \left(\frac{\lambda+\mu}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-\lambda]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(b)-g(t)]^{1-\alpha}} \right]. \end{aligned}$$

The above lemma provides various identities of interest by taking particular values for the parameters λ and μ, out of which we give only a few:

Corollary 2 *With the assumptions of Lemma 1 we have:*
(i) *For any* $x \in (a,b)$,

$$\begin{aligned} I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right) f(x) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(x)]^{1-\alpha}} \right] \end{aligned} \tag{10}$$

and

$$I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right) f(x)$$

$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right]. \quad (11)$$

(ii) *For any* $x \in [a, b]$,

$$\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}f(x)$$
$$+\frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b \frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right]. \quad (12)$$

The proof is obvious by taking $\lambda = \mu = f(x)$ in Lemma 1. These identities were obtained in [14]. If we take in (10)-(12) $x = M_g(a, b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get the corresponding identities were obtained in [14].

Corollary 3 *With the assumptions of Lemma 1 we have:*

$$I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)$$
$$=\frac{1}{\Gamma(\alpha+1)}\left(\left[g(x)-g(a)\right]^{\alpha}f(a)+\left[g(b)-g(x)\right]^{\alpha}f(b)\right)$$
$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \quad (13)$$

and

$$I^{\alpha}_{x-,g}f(a)+I^{\alpha}_{x+,g}f(b)$$
$$=\frac{1}{\Gamma(\alpha+1)}\left(\left[g(x)-g(a)\right]^{\alpha}f(a)+\left[g(b)-g(x)\right]^{\alpha}f(b)\right)$$
$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right], \quad (14)$$

for any $x \in (a, b)$

(ii) *We also have*

$$\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}$$
$$+\frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b \frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right]. \quad (15)$$

The proof of (13) and (14) are obvious by taking $\lambda = f(a)$, $\mu = f(b)$ in Lemma 1. The proof of (15) follows by Lemma 1 on taking $\lambda = f(b)$ and $\mu = f(a)$.

Remark 3 *If we take in* (13) *and* (14) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, *then we get*

$$\begin{aligned}
&I^{\alpha}_{a+,g}f(M_g(a,b)) + I^{\alpha}_{b-,g}f(M_g(a,b)) \\
&\qquad = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{f(a)+f(b)}{2}\right) \\
&+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)[f(t)-f(a)]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)[f(t)-f(b)]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&I^{\alpha}_{M_g(a,b)-,g}f(a) + I^{\alpha}_{M_g(a,b)+,g}f(b) \\
&\qquad = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{f(a)+f(b)}{2}\right) \\
&+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right].
\end{aligned}$$

3 Inequalities for bounded functions

Now, for ϕ, $\Phi \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions, see for instance [15]

$$\begin{aligned}
&\bar{U}_{[a,b]}(\phi,\Phi) \\
&:= \left\{f:[a,b]\to\mathbb{C}|\mathrm{Re}\left[(\Phi-f(t))\left(\overline{f(t)}-\overline{\phi}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}
\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi) := \left\{f:[a,b]\to\mathbb{C}\left|\left|f(t)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for a.e. } t\in[a,b]\right.\right\}.$$

The following representation result may be stated.

Proposition 1 *For any* ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, *we have that* $\bar{U}_{[a,b]}(\phi,\Phi)$ *and* $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\phi,\Phi) = \bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{16}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\mathrm{Re}\left[(\Phi - z)(\bar{z} - \phi)\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| z - \frac{\phi + \Phi}{2} \right|^2 = \mathrm{Re}\left[(\Phi - z)(\bar{z} - \phi)\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (16) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 4 *For any* $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$,*we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\phi, \Phi) = \{ & f : [a,b] \to \mathbb{C} \mid (\mathrm{Re}\,\Phi - \mathrm{Re}\,f(t))(\mathrm{Re}\,f(t) - \mathrm{Re}\,\phi) \\ & + (\mathrm{Im}\,\Phi - \mathrm{Im}\,f(t))(\mathrm{Im}\,f(t) - \mathrm{Im}\,\phi) \geq 0 \text{ for a.e. } t \in [a,b] \}. \end{aligned}$$

Now, if we assume that $\mathrm{Re}(\Phi) \geq \mathrm{Re}(\phi)$ and $\mathrm{Im}(\Phi) \geq \mathrm{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\phi, \Phi) := \{ & f : [a,b] \to \mathbb{C} \mid \mathrm{Re}(\Phi) \geq \mathrm{Re} f(t) \geq \mathrm{Re}(\phi) \\ & \text{and } \mathrm{Im}(\Phi) \geq \mathrm{Im} f(t) \geq \mathrm{Im}(\phi) \text{ for a.e. } t \in [a,b] \}. \end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\phi, \Phi) \subseteq \bar{U}_{[a,b]}(\phi, \Phi).$$

We have:

Theorem 2 *Let* $f : [a,b] \to \mathbb{C}$ *be a complex valued Lebesgue integrable function on the real interval* $[a,b]$, g *be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b) *and* ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$ *such that* $f \in \bar{\Delta}_{[a,b]}(\phi, \Phi)$.

(i) *For any* $x \in (a,b)$,

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{\phi + \Phi}{2\Gamma(\alpha + 1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) \right| \quad (17)$$

$$\le \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right]$$

and

$$\left|I^{\alpha}_{x-,g}f(a)+I^{\alpha}_{x+,g}f(b)-\frac{\phi+\Phi}{2\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\right| \tag{18}$$
$$\le \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right].$$

(ii) *We have*

$$\left|\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{\phi+\Phi}{2}\right| \tag{19}$$
$$\le \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}|\Phi-\phi|[g(b)-g(a)]^{\alpha}.$$

Proof. Using the identity (1) for $\lambda=\mu=\frac{\phi+\Phi}{2}$, we have

$$\begin{aligned} &I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x) \\ &\qquad -\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\frac{\phi+\Phi}{2} \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left[f(t)-\frac{\phi+\Phi}{2}\right]dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b \frac{g'(t)\left[f(t)-\frac{\phi+\Phi}{2}\right]dt}{[g(t)-g(x)]^{1-\alpha}}\right] \end{aligned} \tag{20}$$

for any $x\in(a,b)$.

Taking the modulus in (20), then we get

$$\begin{aligned} &\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\frac{\phi+\Phi}{2}\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left|f(t)-\frac{\phi+\Phi}{2}\right|dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b \frac{g'(t)\left|f(t)-\frac{\phi+\Phi}{2}\right|dt}{[g(t)-g(x)]^{1-\alpha}}\right] \\ &\le \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b \frac{g'(t)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \\ &= \frac{1}{2}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right] \end{aligned}$$

for any $x\in(a,b)$, which proves (17).

The inequality (18) follows in a similar manner from the identity (2).
The inequality (19) follows by (3) for $\lambda = \mu = \frac{\phi+\Phi}{2}$. □

Corollary 5 *With the assumptions of Theorem 2 we have*

$$\left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{\phi+\Phi}{2^{\alpha}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} |\Phi - \phi| \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}$$

and

$$\left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{\phi+\Phi}{2^{\alpha}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} |\Phi - \phi| \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}.$$

Remark 4 *If the function* $f : [a,b] \to \mathbb{R}$ *is measurable and there exists the constants* m, M *such that* $m \leq f(t) \leq M$ *for a.e.* $t \in [a,b]$, *then for any* $x \in (a,b)$ *we have by* (17) *and* (18) *that*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{m+M}{2\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) \right| \leq \frac{1}{2}(M-m) \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right]$$

and

$$\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{m+M}{2\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) \right| \leq \frac{1}{2}(M-m) \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right].$$

In particular,

$$\left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{m+M}{2^{\alpha}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} (M-m) \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}$$

and

$$\left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{m+M}{2^{\alpha}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \right| \leq \frac{1}{2^{\alpha}} (M-m) \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha}.$$

4 Trapezoid inequalities for functions of bounded variation

We have:

Theorem 3 *Let* $f:[a,b]\to\mathbb{C}$ *be a complex valued function of bounded variation on the real interval* $[a,b]$, *and* g *be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b). *Then we have the inequalities*

$$\begin{aligned}
&\left| I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_a^t (f)\, dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_t^b (f)\, dt}{[g(t)-g(x)]^{1-\alpha}} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \end{cases}
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\left| I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_a^t (f)\, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_t^b (f)\, dt}{[g(b)-g(t)]^{1-\alpha}} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \end{cases}
\end{aligned} \tag{22}$$

for any $x\in(a,b)$

(ii) *We also have*

$$\begin{aligned}
&\left|\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
&\leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right] \qquad (23)\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\bigvee_a^b(f).
\end{aligned}$$

Proof. Using the identity (13) and the properties of the modulus, we have

$$\begin{aligned}
&\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{\left[g(x)-g(a)\right]^{\alpha}f(a)+\left[g(b)-g(x)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\left|f(t)-f(a)\right|dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\left|f(t)-f(b)\right|dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right]=:B(x)
\end{aligned}$$

for any $x\in(a,b)$.

Since f is of bounded variation on $[a,b]$, then we have

$$\left|f(t)-f(a)\right|\leq\bigvee_a^t(f)\leq\bigvee_a^x(f)\ \text{ for } a\leq t\leq x$$

and

$$\left|f(t)-f(b)\right|\leq\bigvee_t^b(f)\leq\bigvee_x^b(f)\ \text{ for } x\leq t\leq b.$$

Therefore

$$\begin{aligned}
B(x)&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \\
&\leq\frac{1}{\Gamma(\alpha)}\left[\bigvee_a^x(f)\int_a^x \frac{g'(t)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\bigvee_x^b(f)\int_x^b \frac{g'(t)\,dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \\
&=\frac{1}{\Gamma(\alpha)}\left[\frac{\left(g(x)-g(a)\right)^{\alpha}}{\alpha}\bigvee_a^x(f)+\frac{\left(g(b)-g(x)\right)^{\alpha}}{\alpha}\bigvee_x^b(f)\right] \\
&=\frac{1}{\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}\bigvee_a^x(f)+\left(g(b)-g(x)\right)^{\alpha}\bigvee_x^b(f)\right],
\end{aligned}$$

which proves the first two inequalities in (21).

The last part of (21) is obvious by making use of the elementary Hölder type inequalities for positive real numbers $c,\ d,\ m,\ n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m, n\}(c + d); \\ \\ (m^p + n^p)^{1/p}(c^q + d^q)^{1/q} \text{ with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

The inequality (22) follows in a similar way by utilising the equality (14).

From the equality (15) we have

$$\begin{aligned}
&\left| \frac{I^{\alpha}_{b-,g} f(a) + I^{\alpha}_{a+,g} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \frac{f(b) + f(a)}{2} \right| \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t)\,|f(t) - f(b)|\,dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)\,|f(t) - f(a)|\,dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t) \bigvee_t^b (f)\,dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t) \bigvee_a^t (f)\,dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
&\leq \frac{1}{2\Gamma(\alpha)} \left[\bigvee_a^b (f) \int_a^b \frac{g'(t)\,dt}{[g(b) - g(t)]^{1-\alpha}} + \bigvee_a^b (f) \int_a^b \frac{g'(t)\,dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\
&= \frac{1}{2\Gamma(\alpha)} \left[\bigvee_a^b (f) \frac{[g(b) - g(a)]^{\alpha}}{\alpha} + \bigvee_a^b (f) \frac{[g(b) - g(a)]^{\alpha}}{\alpha} \right] \\
&= \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \bigvee_a^b (f),
\end{aligned}$$

which proves (23). □

Corollary 6 *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
&\left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{f(a) + f(b)}{2^{\alpha}\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_a^t (f)\,dt}{[g(M_g(a,b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_t^b (f)\,dt}{[g(t) - g(M_g(a,b))]^{1-\alpha}} \right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (g(b) - g(a))^{\alpha} \bigvee_a^b (f)
\end{aligned}$$

and

$$\left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_a^t (f)\, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_t^b (f)\, dt}{[g(b)-g(t)]^{1-\alpha}} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (g(b)-g(a))^{\alpha} \bigvee_a^b (f).$$

5 Inequalities for Hölder's continuous functions

We say that the function $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$ if

$$|f(t) - f(s)| \leq H |t-s|^r \tag{24}$$

for any $t, s \in [a,b]$. If $r = 1$ and $H = L$ we call the function L-*Lipschitzian* on $[a,b]$.

Theorem 4 *Assume that* $f : [a,b] \to \mathbb{C}$ *is* r-H-*Hölder continuous on* $[a,b]$ *with* $r \in (0,1]$ *and* $H > 0$, *and* g *be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b). *Then*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right|$$
$$\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(t-a)^r\, dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)(b-t)^r\, dt}{[g(t)-g(x)]^{1-\alpha}} \right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} [(g(x)-g(a))^{\alpha}(x-a)^r + (g(b)-g(x))^{\alpha}(b-x)^r]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left|g(x) - \frac{g(a)+g(b)}{2}\right|\right]^{\alpha} [(x-a)^r + (b-x)^r]; \\ ((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p})^{1/p} ((x-a)^{rq} + (b-x)^{rq})^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\ ((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}) \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \end{cases} \tag{25}$$

and

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$
$$\le\frac{H}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)(t-a)^r dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b\frac{g'(t)(b-t)^r dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$
$$\le\frac{H}{\Gamma(\alpha+1)}[(g(x)-g(a))^{\alpha}(x-a)^r+(g(b)-g(x))^{\alpha}(b-x)^r]$$
$$\le\frac{H}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}[(x-a)^r+(b-x)^r];\\ ((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p})^{1/p}((x-a)^{rq}+(b-x)^{rq})^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ ((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha})\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\end{cases}\tag{26}$$

for any $x\in(a,b)$

(ii) *We also have*

$$\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right|$$
$$\le\frac{H}{2\Gamma(\alpha)}\left[\int_a^b\frac{g'(t)(b-t)^r dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b\frac{g'(t)(t-a)^r dt}{[g(t)-g(a)]^{1-\alpha}}\right]\tag{27}$$
$$\le\frac{H}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}(b-a)^r.$$

Proof. Using the identity (13) and the properties of the modulus, we have

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)|f(t)-f(a)|dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)|f(t)-f(b)|dt}{[g(t)-g(x)]^{1-\alpha}}\right]=:C(x)$$

for any $x\in(a,b)$.

Since $f:[a,b]\to\mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $H>0$, hence

$$C(x)\le\frac{H}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)(t-a)^r dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)(b-t)^r dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$

$$\begin{aligned}
&\leq \frac{H}{\Gamma(\alpha)}\left[(x-a)^r\int_a^x \frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}}+(b-x)^r\int_x^b \frac{g'(t)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]\\
&= \frac{H}{\Gamma(\alpha)}\left[(x-a)^r\frac{(g(x)-g(a))^{\alpha}}{\alpha}+(b-x)^r\frac{(g(b)-g(x))^{\alpha}}{\alpha}\right]\\
&= \frac{H}{\Gamma(\alpha+1)}\left[(x-a)^r(g(x)-g(a))^{\alpha}+(b-x)^r(g(b)-g(x))^{\alpha}\right],
\end{aligned}$$

for any $x \in (a,b)$, which proves the first two inequalities in (25). The rest is obvious.

The inequality (26) follows in a similar way by utilising the equality (14).

The inequality (27) follows by utilising the equality (15). □

Corollary 7 *With the assumptions of Theorem 4 we have*

$$\begin{aligned}
&\left|I^{\alpha}_{a+,g}f(M_g(a,b))+I^{\alpha}_{b-,g}f(M_g(a,b))-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right|\\
&\leq \frac{H}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)(t-a)^r\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)(b-t)^r\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]\\
&\leq \frac{H}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\left[(M_g(a,b)-a)^r+(b-M_g(a,b))^r\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right|\\
&\leq \frac{H}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)(t-a)^r\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)(b-t)^r\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]\\
&\leq \frac{H}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\left[(M_g(a,b)-a)^r+(b-M_g(a,b))^r\right].
\end{aligned}$$

6 Applications for Hadamard fractional integrals

If we take $g(t)=\ln t$ and $0 \leq a < x \leq b$, then by Theorem 3 for Hadamard fractional integrals H^{α}_{a+} and H^{α}_{b-} we have for $f:[a,b]\to\mathbb{C}$, a function of bounded variation on $[a,b]$ that

$$\left|H^{\alpha}_{a+}f(x)+H^{\alpha}_{b-}f(x)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\bigvee_a^t(f)\,dt}{t}+\int_x^b \frac{\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}\bigvee_a^x(f)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}\bigvee_x^b(f)\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases} \tag{28}$$

and

$$\left|H_{x-}^{\alpha}f(a)+H_{x+}^{\alpha}f(b)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\bigvee_a^t(f)\,dt}{t}+\int_x^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left[\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}\bigvee_a^x(f)+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\bigvee_x^b(f)\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases} \tag{29}$$

for any $x \in (a, b)$

We also have

$$\left|\frac{H_{b-}^{\alpha}f(a)+H_{a+}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right|$$
$$\leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}+\int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}g'(t)\bigvee_a^t(f)\,dt}{t}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\bigvee_a^b(f).$$

If we take in (28) and (29) $x = G(a,b)$, then we get

$$\begin{aligned}
&\left| H^{\alpha}_{a+} f(G(a,b)) + H^{\alpha}_{b-} f(G(a,b)) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} \bigvee_a^t (f)\, dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t} \right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b (f)
\end{aligned}$$

and

$$\begin{aligned}
&\left| H^{\alpha}_{G(a,b)-} f(a) + H^{\alpha}_{G(a,b)+} f(b) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} \bigvee_a^t (f)\, dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t} \right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b (f).
\end{aligned}$$

Assume that $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$. If we take $g(t) = \ln t$ and $0 \leq a < x \leq b$ in Theorem 4, then we get

$$\begin{aligned}
&\left| H^{\alpha}_{a+} f(x) + H^{\alpha}_{b-} f(x) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
&\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} (t-a)^r\, dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} (b-t)^r\, dt}{t} \right] \\
&\leq \frac{H}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} (x-a)^r + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} (b-x)^r \right] \\
&\leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b (f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p} \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right) \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases}
\end{aligned}$$

and

$$\left| H^{\alpha}_{x-}f(a) + H^{\alpha}_{x+}f(b) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right|$$

$$\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} (t-a)^r \, dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} (b-t)^r \, dt}{t} \right]$$

$$\leq \frac{H}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} (x-a)^r + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} (b-x)^r \right]$$

$$\leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b(f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p} \left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right) \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right] \end{cases} \tag{30}$$

for any $x \in (a, b)$.

We also have

$$\left| \frac{H^{\alpha}_{b-}f(a) + H^{\alpha}_{a+}f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \frac{f(b) + f(a)}{2} \right|$$

$$\leq \frac{H}{2\Gamma(\alpha)} \left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} (b-t)^r \, dt}{t} + \int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} (t-a)^r \, dt}{t} \right] \tag{31}$$

$$\leq \frac{H}{\Gamma(\alpha+1)} (b-a)^r \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}.$$

If we take in (30) and (31) $x = G(a, b)$, then we get

$$\left| H^{\alpha}_{a+}f(G(a,b)) + H^{\alpha}_{b-}f(G(a,b)) - \frac{f(a) + f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right|$$

$$\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} (t-a)^r dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} (b-t)^r dt}{t} \right]$$

$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} (b-a)^r$$

and

$$\left| H^{\alpha}_{G(a,b)-}f(a) + H^{\alpha}_{G(a,b)+}f(b) - \frac{f(a) + f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{G(a,b)} \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}(t-a)^{r}\,dt}{t} + \int_{G(a,b)}^{b} \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}(b-t)^{r}\,dt}{t}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}(b-a)^{r}.$$

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